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Nondegeneracy and uniqueness for boundary blow-up elliptic problems ☆

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Abstract

In this paper, we use for the first time linearization techniques to deal with boundary blow-up elliptic problems. After introducing a convenient functional setting, we show that the problem $\Delta u = \lambda a(x)u^p + g(x, u)$ in Ω, with $u = +\infty$ on $\partial\Omega$, has a unique positive solution for large enough λ, and determine its asymptotic behavior as $\lambda \to +\infty$. Here $p > 1$, $a(x)$ is a continuous function which can be singular near $\partial\Omega$ and $g(x, u)$ is a perturbation term with potential growth near zero and infinity. We also consider more general problems, obtained by replacing u^p by e^u or a "logistic type" function $f(u)$.

Keywords: Boundary blow-up problems; Nondegeneracy; Implicit function theorem

1. Introduction

Problems with blow-up at the boundary have been largely studied over the years. Their origin seems to be the work by Bieberbach [7], where they appear in the study of automorphic functions in the plane and of Riemannian surfaces with constant negative curvature. They also arise when analyzing the equilibrium of a charged gas in a container (cf. [24]) or in population dynamics, when the logistic equation with refuge is considered (see [15,18]).

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After [7], an enormous amount of works have dealt with these problems, mainly concerned with the issues of existence, uniqueness and behavior near the boundary for positive solutions, both for single equations (see [1,3–6,8,9,11,12,15,17,19,25–30,32–36]) and lately for systems (cf. [10,13,14,20–22,31]).

We also mention that there have been some recent applications of this kind of problems, for instance to Liouville theorems for logistic-like equations in $\mathbb{R}^N$ in [16] or to the analysis of blow-up for a parabolic equation with a nonlinear boundary condition in [2].

Among all the boundary blow-up elliptic problems, one of them seems to be best understood, namely

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a(x)$ is a continuous weight function, $p > 1$ and Ω is a bounded smooth domain of $\mathbb{R}^N$. The boundary condition is meant as $u(x) \to +\infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0+$. Problem (1.1) has been studied for instance in [4,5,8,9,11,12,19,26,27,30,33,35,36]. It is known that for very general weight functions $a(x)$, which can be even singular near $\partial\Omega$, there exists a unique solution to (1.1).

Our aim here is to show uniqueness for a slightly more general class of problems related to (1.1). Specifically, we are going to consider

$$\begin{cases} \Delta u = \lambda a(x)u^p + g(x,u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g(x,u)$ is a prescribed continuous function. For the weight function $a(x)$ we are assuming that there exists $\gamma \in \mathbb{R}$ and positive constants C_1, C_2 such that

$$C_1 d(x)^{-\gamma} \leqslant a(x) \leqslant C_2 d(x)^{-\gamma}, \quad x \in \Omega. \tag{A}$$

Since it is shown in [9] that problem (1.1) cannot have positive solutions when $\gamma \geqslant 2$, we are assuming throughout the paper that $\gamma < 2$.

We remark at this point that uniqueness of solutions to boundary blow-up elliptic problems has been obtained frequently in the literature by means of boundary estimates: it is proved first that the quotient $u(x)/v(x)$ of any two solutions u, v approaches one as $d(x)$ tends to zero, and then some sort of monotonicity of the nonlinearity gives uniqueness. This approach fails for problems such as (1.1) if the weight $a(x)$ does not have a prescribed behavior near $\partial\Omega$ (observe that our hypotheses allow $a(x)$ to be oscillating near $\partial\Omega$). The other possible method to prove uniqueness is the one used in [9,26], which can be only applied to power-like nonlinearities, and global bounds for the solutions in terms of $d(x)$ are required, rather than estimates near the boundary. In Section 3, we generalize the proof in [9] to obtain a new uniqueness result for problems like (1.1), with u^p replaced by more general nonlinearities $f(u)$ which include $u^p - bu$, or $u^p + bu^q$, $b > 0$, $1 \leqslant q < p$.

But when dealing with general problems which have both failures, namely, the weights can oscillate near $\partial\Omega$ and the nonlinearities do not have a convenient monotonicity, there are no methods available at the moment for proving uniqueness. We are introducing in the present work a functional setting, which allows us to use continuation arguments for boundary blow-up problems.

The success of our approach relies in knowing the growth of the solutions. For nonlinearities which behave like a power near infinity, one can typically deduce that all positive solutions u verify $C_1 d(x)^{-\alpha} \leqslant u(x) \leqslant C_2 d(x)^{-\alpha}$ in Ω, for some positive constants C_1, C_2 and some $\alpha > 0$. Thus a reasonable way to proceed is to consider a space of functions u such that $\sup_\Omega d(x)^\alpha |u(x)| < +\infty$, which in particular contains all solutions. We are showing that this type of spaces is very natural for problems like (1.2).

Our main tool is the implicit function theorem. However, it is worth saying that when proving nondegeneracy of a solution we are facing a problem with no boundary conditions, and thus usual tools like maximum principles are not useful. Also, since compactness is not easy to achieve for our operators, we cannot use a Fredholm alternative, but this can be solved with ad hoc methods.

We now state our results. It turns out that the best way to study (1.2) is to perform the scaling $u = \lambda^{-\frac{1}{p-1}} v$ to arrive at the problem

$$\begin{cases} \Delta v = a(x) v^p + \lambda^{\frac{1}{p-1}} g\left(x, \lambda^{-\frac{1}{p-1}} v\right) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

If we assume the growth condition $|g(x,u)| \leqslant C d(x)^{-\gamma} u^p$ for $u > 0$ on g, problem (1.3) is a small perturbation of (1.1) for large λ. Thus, it can be expected that at least for large λ there is a unique positive solution for this problem. This is indeed the situation, if we impose some other (technical) hypotheses on g.

Theorem 1. *Assume a is continuous and verifies hypotheses* (A), *and $g : \Omega \times [0, +\infty) \to \mathbb{R}$ is a continuous function such that the second derivative with respect to u is continuous and*

(i) $g(x,0) = 0,\ g'(x,0) = 0$;
(ii) $|g''(x,u)| \leqslant C d(x)^{-\gamma} u^{p-2}$ *for* $x \in \Omega,\ u > 0$.

Then there exists $\lambda_0 > 0$ such that problem (1.2) *has a unique positive strong solution $u_\lambda \in C^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ for $\lambda \geqslant \lambda_0$. Moreover,*

$$\lim_{\lambda \to +\infty} \sup_\Omega d(x)^\alpha |\lambda^{\frac{1}{p-1}} u_\lambda(x) - U(x)| = 0,$$

where U is the unique solution to (1.1), *and $\alpha = (2-\gamma)/(p-1)$. In particular, there exist positive constants C and C' not depending on λ such that*

$$C \lambda^{-\frac{1}{p-1}} d(x)^{-\alpha} \leqslant u_\lambda(x) \leqslant C' \lambda^{-\frac{1}{p-1}} d(x)^{-\alpha}$$

in Ω.

Theorem 1 can be obtained as a byproduct of the more general result in Theorem 11 of Section 4.1, which is valid for the following class of problems:

$$\begin{cases} \Delta v = a(x)v^p + h(x, v, \varepsilon) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where h is a perturbation which vanishes for $\varepsilon = 0$, and verifies some growth assumptions (see hypotheses (H) in §4.1).

Although, our approach could seem to be limited to deal with problems which involve nonlinearities which behave like powers at infinity, we find that with a convenient change of variables we can also treat exponential problems like

$$\begin{cases} \Delta u = \lambda a(x)e^u + g(x, u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

and we consequently obtain a similar result to Theorem 1, which is a byproduct of the more general Theorem 13 in Section 4.2.

Theorem 2. *Assume a is continuous and verifies hypotheses* (A), *and* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that the second derivative with respect to u is continuous and*

(i) $\lim_{u\to-\infty} g(x, u) = 0$, $\lim_{u\to-\infty} g'(x, u) = 0$;
(ii) $|g''(x, u)| \leqslant Cd(x)^{-\gamma}e^u$ *for* $x \in \Omega$, $u \in \mathbb{R}$.

Then there exists $\lambda_0 > 0$ *such that problem* (1.5) *has a unique strong solution* $u_\lambda \in C^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ *for* $\lambda \geqslant \lambda_0$. *Moreover,*

$$\lim_{\lambda\to+\infty} \sup_{\Omega} d(x)^{2-\gamma} |\lambda e^{u_\lambda(x)} - e^{V(x)}| = 0,$$

where V is the unique solution to (1.5) *with* $\lambda = 1$ *and* $g \equiv 0$. *In particular, there exist positive constants C and* C' *not depending on* λ *such that*

$$C\lambda^{-1}d(x)^{\gamma-2} \leqslant e^{u_\lambda(x)} \leqslant C'\lambda^{-1}d(x)^{\gamma-2}$$

in Ω.

We finally remark that with the same ideas, some more general perturbed problems can (and will) be considered, for instance

$$\begin{cases} \Delta v = a(x)f(v) + h(x, v, \varepsilon) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where the nonlinearity $f(u)$ is a logistic-type function, like $u^p - bu$ or $u^p + bu^q$, $b > 0$, $1 \leqslant q < p$. The corresponding Theorem will be stated in Section 4.3.

The paper is organized as follows: in Section 2 we introduce the spaces where our problems will be considered, and treat some operators on them. Section 3 is devoted to recall some facts for the unperturbed problems (1.2) and (1.5) with $g \equiv 0$, and to prove a new uniqueness result for (1.6) with $h \equiv 0$ (Theorem 10). Finally, in Section 4 we show the nondegeneracy of solutions, and prove Theorems 11, 13 and 14, which will in particular imply Theorems 1 and 2.

2. Functional framework

In this section, we introduce the spaces we are using throughout the paper, together with some differential and Nemytskii-type operators defined on them. The reader will notice that some other choices of spaces are possible, such as subspaces of locally Hölder continuous functions, but we have preferred to work in a "strong" setting in order to keep technical details to a minimum.

Fix a real number $\alpha \geqslant -2$, and define

$$Y_\alpha := \{u \in L^\infty_{\mathrm{loc}}(\Omega): \ \|u\|_Y < +\infty\},$$

$$X_\alpha := \{u \in C^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega): \ \|u\|_X < +\infty\},$$

where

$$\|u\|_Y = \sup_{x\in\Omega} d(x)^{\alpha+2}|u(x)|,$$

$$\|u\|_X = \sup_{\Omega} d(x)^{\alpha}|u(x)| + \max_{1\leqslant i\leqslant N} \sup_{\Omega} d(x)^{\alpha+1}|\partial_i u(x)| + \sup_{\Omega} d(x)^{\alpha+2}|\Delta u(x)|.$$

Here and in what follows all supremums are understood to be essential. We will also briefly deal with the space $Y_{\alpha-2}$, whose norm will be denoted by $\|\cdot\|$.

The first important remark is that the spaces X_α and Y_α are Banach spaces when endowed with their respective norms. This is not at all straightforward for the space X_α.

Lemma 3. *The spaces X_α and Y_α are Banach spaces when provided with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in Y_α. Since $d(x)$ is bounded away from zero on compacts of Ω, it follows that there exists a locally bounded function u such that $u_n \to u$ in $L^\infty_{\mathrm{loc}}(\Omega)$. Passing to the limit in $|u_n(x)| \leqslant \sup_{n\in\mathbb{N}} \|u_n\|_Y d(x)^{-\alpha-2}$ we obtain $u \in Y_\alpha$.

We claim that $u_n \to u$ in Y_α. To show this, let $\varepsilon > 0$, and take $x \in \Omega$. Then $u_n(x) \to u(x)$, so we can fix m so large that $|u_m(x) - u(x)| \leqslant \varepsilon$. Thus

$$d(x)^{\alpha+2}|u_n(x) - u(x)| \leqslant \|u_n - u_m\|_Y + \left(\sup_\Omega d\right)^{\alpha+2} \varepsilon \leqslant C\varepsilon$$

for a positive constant C, if n and m are large enough. Since this inequality is valid, a.e., taking sup we arrive at $u_n \to u$ in Y_α.

Now let $\{u_n\}$ be a Cauchy sequence in X_α. By a similar reasoning as before, it follows that there exist continuous functions u, f_i, $1 \leqslant i \leqslant N$ and $g \in L^\infty_{\mathrm{loc}}(\Omega)$ such that

$$\begin{aligned}
&\sup_\Omega d(x)^\alpha |u_n(x) - u(x)| \to 0, \\
&\sup_\Omega d(x)^{\alpha+1} |\partial_i u_n(x) - f_i(x)| \to 0, \\
&\sup_\Omega d(x)^{\alpha+2} |\Delta u_n(x) - g(x)| \to 0.
\end{aligned} \tag{2.1}$$

It follows that $f_i = \partial_i u$ in the weak sense, for if $\phi \in C_0^\infty(\Omega)$, then

$$\int_\Omega \partial_i u_n \phi = -\int_\Omega u_n \partial_i \phi \to -\int_\Omega u \partial_i \phi.$$

In particular, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, and it can be similarly shown that $\Delta u = g$ in the weak sense. Thus standard regularity theory (see [23, Chapter 8]) gives that $u \in H^2_{\mathrm{loc}}(\Omega)$ and $\Delta u = g$ a.e. Since $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$, it also follows from [23, Chapter 9], that $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$ for every $q > 1$. Hence taking $q > N$ the Sobolev embedding implies $u \in C^1(\Omega)$, and thus $u \in X_\alpha$. By (2.1) $u_n \to u$ in X_α, and X_α is a Banach space. $\square$

We now turn to consider some important operators defined on X_α with values in Y_α. The first one is the Laplacian: by definition it follows that if $u \in X_\alpha$, then $\Delta u \in Y_\alpha$, and $\|\Delta u\|_Y \leqslant \|u\|_X$. Thus Δ is a bounded linear operator from X_α to Y_α.

Next, we will study the Nemytskii operators associated to functions $f(x,u)$, i.e. operators $F : Y_{\alpha-2} \to Y_\alpha$ defined as $F(u) = f(x, u(x))$. As in the case of L^p spaces, we have to restrict the growth of the function f at infinity. Observe that our hypotheses below allow $f(\cdot, u)$ to be singular at $\partial\Omega$, for fixed u, and that the growth restriction on $f''(x,u)$ in (c) automatically implies the corresponding growth for $f'(x,u)$ and $f(x,u)$ in (a) and (b), respectively. We remark that the space X_α is continuously embedded in $Y_{\alpha-2}$.

Theorem 4. *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. Let $\gamma, p \in \mathbb{R}$ such that $p > 1$ and $\gamma \leqslant 2 - \alpha(p-1)$. Then*

(a) *If $|f(x,u)| \leqslant C d(x)^{-\gamma}(1+|u|)^p$ for some $C > 0$, then $F : Y_{\alpha-2} \to Y_\alpha$ is well-defined.*
(b) *If the derivative of f with respect to u exists and $|f'(x,u)| \leqslant C d(x)^{-\gamma}(1+|u|)^{p-1}$, then F is locally Lipschitz-continuous.*
(c) *If the second derivative of f with respect to u exists and $|f''(x,u)| \leqslant C d(x)^{-\gamma}(1+|u|)^{p-2}$, then F is Fréchet differentiable at any u such that $\inf_\Omega d(x)^\alpha |u(x)| > 0$,*

and

$$F'(u)\varphi = f'(x,u)\varphi.$$

Moreover, F is C^1 in a neighborhood of u.

Proof. Let $u \in Y_{\alpha-2}$. Then if C denotes a generic constant:

$$d^{\alpha+2}|f(x,u)| \leqslant C d^{\alpha+2-\gamma}(1+|u|)^p \leqslant C(1+\|u\|)^p$$

since $\gamma \leqslant 2-\alpha(p-1)$. This shows (a). To prove (b), let $u, v \in Y_{\alpha-2}$, and apply the mean value theorem to obtain

$$d^{\alpha+2}|f(x,u)-f(x,v)| = d^{\alpha+2}|f'(x,\xi)||u-v| \leqslant C d^{2-\gamma}(1+|\xi|)^{p-1}\|u-v\|$$

for some intermediate value $\xi(x)$ comprised between $u(x)$ and $v(x)$. If u and v are in a bounded set of $Y_{\alpha-2}$, say $\|u\|, \|v\| \leqslant M$, then $|\xi| \leqslant M d^{-\alpha}$ and so

$$d^{\alpha+2}|f(x,u)-f(x,v)| \leqslant C d^{2-\gamma-\alpha(p-1)}\|u-v\|,$$

as was to be proved.

We now show (c). Assume $\inf_\Omega d(x)^\alpha |u(x)| > 0$. Then there exists $\delta > 0$ such that $\inf_\Omega d(x)^\alpha |(u+\varphi)(x)| \geqslant \delta$ for $\|\varphi\|$ sufficiently small. Thus, using the Taylor expansion for $f(x,\cdot)$ up to the second order

$$\begin{aligned} d^{\alpha+2}|f(x,u+\varphi)-f(x,u)-f'(x,u)\varphi| &= \tfrac{1}{2} d^{\alpha+2}|f''(x,\xi)||\varphi|^2 \\ &\leqslant C d^{2-\gamma-\alpha}(1+|\xi|)^{p-2}\|\varphi\|^2. \end{aligned}$$

If $p \geqslant 2$, we conclude as before. If on the contrary $1 < p < 2$, observing that $|\xi| \geqslant \delta d^{-\alpha}$, we have

$$d^{\alpha+2}|f(x,u+\varphi)-f(x,u)-f'(x,u)\varphi| \leqslant C d^{2-\gamma-\alpha(p-1)}\|\varphi\|^2$$

which proves the differentiability of F. Finally, note that F is also differentiable in a neighborhood of u, since the condition $\inf_\Omega d(x)^\alpha |v(x)| > 0$ holds for every v close to u in $Y_{\alpha-2}$. Proceeding as before we obtain

$$d^{\alpha+2}|f'(x,u)\varphi - f'(x,v)\varphi| \leqslant C\|u-v\|\|\varphi\|$$

for every $\varphi \in Y_{\alpha-2}$, and this proves that F' is (Lipschitz) continuous in a neighborhood of u. $\square$

Remark 1. As the above proof shows, the condition $\inf_\Omega d(x)^\alpha |u(x)| > 0$ is unnecessary if $p \geqslant 2$.

The next corollary is a direct consequence of Theorem 4, taking into account that $X_\alpha \subset Y_{\alpha-2}$ continuously and the fact that Δ is a bounded linear operator from X_α to Y_α.

Corollary 5. *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and that the second derivative of f with respect to u is continuous, with $|f''(x,u)| \leqslant C d(x)^{-\gamma}(1+|u|)^{p-2}$, and $\gamma \leqslant 2 - \alpha(p-1)$. Then the operator $T : X_\alpha \to Y_\alpha$ defined as $T(u) = \Delta u - f(x,u)$ is C^1 in a neighborhood of any u such that $\inf_\Omega d(x)^\alpha |u(x)| > 0$, and*

$$T'(u)\varphi = \Delta\varphi - f'(x,u)\varphi$$

for every $\varphi \in X_\alpha$.

We finally consider a nonlinear differential operator which will be needed when treating nonlinearities of exponential growth. We define $D(Q) = \{u \in X_\alpha : \inf_\Omega d(x)^\alpha |u(x)| > 0\}$, and $Q : D(Q) \to Y_\alpha$ by

$$Q(u) = \frac{|\nabla u|^2}{u}.$$

It is easily seen that Q is well defined. We also have

Theorem 6. *Let $u \in D(Q)$. Then the operator Q is C^1 in a neighborhood of u and*

$$Q'(u)\varphi = \frac{2}{u}\nabla u \nabla\varphi - \frac{|\nabla u|^2}{u^2}\varphi \tag{2.2}$$

for every $\varphi \in X_\alpha$.

Proof. We first show that the operator in the right-hand side of (2.2) is bounded. Let $\delta_0 = \inf_\Omega d(x)^\alpha |u(x)|$. Then

$$d(x)^{\alpha+2}\left|\frac{2}{u}\nabla u \nabla\varphi - \frac{|\nabla u|^2}{u^2}\varphi\right| \leqslant \left(\frac{2}{\delta_0}\|u\|_X + \frac{1}{\delta_0^2}\|u\|_X^2\right)\|\varphi\|_X$$

for every $x \in \Omega$. Taking sup, we arrive at the desired inequality.

To see that the operator defined by (2.2) is indeed the Fréchet derivative of Q, notice that if $\|\varphi\|_X$ is small, then there exists $\delta > 0$ such that $d(x)^\alpha |u + \varphi|(x) \geqslant \delta$ in Ω.

Hence

$$
\begin{aligned}
&d^{\alpha+2}|Q(u+\varphi)-Q(u)-Q'(u)\varphi| \\
&\quad = d^{\alpha+2}\left|\frac{|\nabla(u+\varphi)|^2}{u+\varphi}-\frac{|\nabla u|^2}{u}-\frac{2}{u}\nabla u\nabla\varphi+\frac{|\nabla u|^2}{u^2}\varphi\right| \\
&\quad = d^{\alpha+2}\left||\nabla u|^2\frac{\varphi^2}{u^2(u+\varphi)}-2\nabla u\nabla\varphi\frac{\varphi}{u(u+\varphi)}+\frac{|\nabla\varphi|^2}{u+\varphi}\right| \\
&\quad \leqslant \left(\frac{\|u\|_X^2}{\delta^3}+\frac{2\|u\|_X}{\delta^2}+\frac{1}{\delta}\right)\|\varphi\|_X^2.
\end{aligned}
$$

Taking sup proves the differentiability of Q. Finally, if $u, v \in D(Q)$ and $v \to u$,

$$
\begin{aligned}
d^{\alpha+2}|Q'(u)\varphi-Q'(v)\varphi| &\leqslant \left(d\left|\frac{2}{u}\nabla u-\frac{2}{v}\nabla v\right|+d^2\left|\frac{|\nabla u|^2}{u^2}-\frac{|\nabla v|^2}{v^2}\right|\right)\|\varphi\|_X \\
&\leqslant C\left(d^{1+2\alpha}|v\nabla u-u\nabla v|+d^{2+4\alpha}|u^2|\nabla v|^2-v^2|\nabla u|^2|\right)\|\varphi\|_X \\
&\leqslant C\|u-v\|_X\|\varphi\|_X,
\end{aligned}
$$

which proves the (Lipschitz) continuity of Q'. This concludes the proof. □

Corollary 7. *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and that the second derivative of f with respect to u is continuous, with $|f''(x,u)| \leqslant Cd(x)^{-\gamma}(1+|u|)^{p-2}$, and $\gamma \leqslant 2-\alpha(p-1)$. Then the operator $S : D(Q) \to Y_\alpha$ defined as $S(u) = \Delta u - Q(u) - f(x,u)$ is C^1 in a neighborhood of any $u \in D(Q)$, and*

$$
S'(u)\varphi = \Delta\varphi - \frac{2}{u}\nabla u\nabla\varphi + \frac{|\nabla u|^2}{u^2}\varphi - f'(x,u)\varphi
$$

for every $\varphi \in X_\alpha$.

3. Uniqueness of unperturbed problems

This section is devoted to the study of the unperturbed problems quoted in the introduction. We begin by recalling some facts regarding the problems with power and exponential nonlinearities, which are already known. Thus we consider

$$
\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{3.1}
$$

for $p > 1$ and

$$\begin{cases} \Delta u = a(x)e^u & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The following results are part of Theorems 1 and 2 in [9] (solutions in $C^{2,\eta}(\Omega)$ were considered there, since the weight $a(x)$ was assumed to be locally Hölder continuous; the adaptation to the strong setting is straightforward). We especially remark the global bounds on the solutions in terms of negative powers of $d(x)$, which in Section 4 will prove to be essential. Recall that we are assuming $\gamma < 2$.

Theorem 8. *Assume a is continuous and verifies hypotheses* (A). *Then problem* (3.1) *has a unique positive solution $U \in C^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$. Moreover, there exist positive constants D_1, D_2 such that $D_1 d(x)^{-\alpha} \leqslant U(x) \leqslant D_2 d(x)^{-\alpha}$ in Ω, where $\alpha = (2-\gamma)/(p-1)$.*

Theorem 9. *Assume a is continuous and verifies hypotheses* (A). *Then problem* (3.2) *has a unique solution $V \in C^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$. Moreover, there exist positive constants D'_1, D'_2 such that $D'_1 d(x)^{\gamma-2} \leqslant e^{V(x)} \leqslant D'_2 d(x)^{\gamma-2}$ in Ω.*

Next we are stating and proving a new uniqueness result for the more general problem

$$\begin{cases} \Delta u = a(x) f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where we assume that $a(x)$ is a continuous function verifying (A) in the introduction and f verifies the following hypotheses:

(a) $f : (0, +\infty) \to \mathbb{R}$ is continuous;
(b) $f(u)/u$ is increasing for $u > 0$;
(c) $\lim_{u\to+\infty} f(u)/u^p = 1$.

As already quoted, an important special case is the "logistic-type" nonlinearity $f(u) = u^p - bu$, for $b \in \mathbb{R}$. It is worth noting that the argument used in [19] to prove uniqueness is not valid here, since the behavior of positive solutions near $\partial\Omega$ is not expected to be determinable. For problem (3.3) we have the following existence and uniqueness result:

Theorem 10. *Assume a is continuous and verifies hypotheses* (A), *while f verifies* (a)–(c). *Then problem* (3.3) *has a unique positive solution Z. Moreover, there exist positive constants D_1, D_2 such that $D_1 d(x)^{-\alpha} \leqslant Z(x) \leqslant D_2 d(x)^{-\alpha}$ in Ω, where $\alpha = (2-\gamma)/(p-1)$.*

Before proceeding to prove Theorem 10, some preliminary remarks are in order. Notice that if $f(u)/u$ is increasing, then f can have at most a positive zero u_0 (if no such zero exists, we let $u_0 = 0$). If u is a positive solution to $\Delta u = a(x)f(u)$ in Ω, and x_0 is a point where u reaches its minimum, it follows that $f(u(x_0)) \geqslant 0$, and thus $u \geqslant u_0$. Hence by the strong maximum principle, $u(x) > u_0$ for every $x \in \Omega$. We also deduce then that f is increasing in $[\min u, +\infty)$.

On the other hand hypothesis (b) implies that for $k > 1$ and $u > u_0$, $f(ku) > kf(u)$. Thus by (a) and (c), for $k_0 > 1$ and $\bar{u} > u_0$ fixed, there exists $\theta > 1$ such that $f(ku) \geqslant k\theta f(u)$ for every $k \geqslant k_0$ and $u \geqslant \bar{u}$. Notice also that by (c), $C_1 u^p \leqslant f(u) \leqslant C_2 u^p$ in $[\min u, +\infty)$ for certain positive constants C_1 and C_2.

Proof of Theorem 10. The existence of solutions is a consequence of Theorem 1 in [17]. Let us see that all positive solutions u verify $D_1 d(x)^{-\alpha} \leqslant u(x) \leqslant D_2 d(x)^{-\alpha}$ in Ω, for some positive constants D_1, D_2. Indeed, since $\Delta u = a(x)f(u) \leqslant C_2 a(x) u^p$ in Ω, the method of sub and supersolutions (cf. [19, Lemma 4], [17, Lemma 1]) and the uniqueness given by Theorem 8 imply $u \geqslant C_2^{-\frac{1}{p-1}} U$ in Ω, and similarly $u \leqslant C_1^{-\frac{1}{p-1}} U$. This proves the estimates. In particular, if u and v are positive solutions, the quotient u/v is bounded and bounded away from zero in Ω.

We now turn to prove uniqueness. We adapt the argument in [9,26]. Let u, v be positive solutions to (3.3), and assume there exist $x_0 \in \Omega$, $k > 1$ such that $u(x_0) > kv(x_0)$. Define $\Omega_0 = \{u > kv\} \cap B_r(x_0)$, where $r = d(x_0)/2$. Then

$$\Delta u = a(x)f(u) > a(x)f(kv) \geqslant \theta a(x) k f(v)$$

in Ω_0, and so $\Delta(u - kv) > a(x)(\theta - 1)kf(v) \geqslant Ckd(x)^{-\gamma} v^p$ in Ω_0. Using the lower estimate for v we moreover have $\Delta(u - kv) > Ckr^{-\alpha p - \gamma}$ in Ω_0. Thus, setting $w(x) = Ckr^{-\alpha p-\gamma}(r^2 - |x - x_0|^2)/2N$, we obtain $\Delta(u - kv + w) > 0$ in Ω_0. By the maximum principle, there exists a point $x_1 \in \partial\Omega_0$ such that $w(x_0) < (u - kv + w)(x_0) < (u - kv + w)(x_1)$. It easily follows that $x_1 \in \partial B_r(x_0)$, since otherwise $w(x_0) < w(x_1)$, which is not possible. Thus,

$$u(x_1) - kv(x_1) > \frac{Ck}{2N} r^{-\alpha} \geqslant Ckv(x_1).$$

We iterate this procedure to obtain a sequence of points $\{x_n\} \subset \Omega$ such that $u(x_n) > (1+C)^n kv(x_n)$. But this contradicts the boundedness of u/v. Thus $u \leqslant v$, and symmetrically $u \geqslant v$, which shows uniqueness. □

4. Nondegeneracy: Uniqueness

In this section, we analyze problems (1.2) and (1.5). Our main ingredient is the implicit function theorem applied in the spaces introduced in Section 2.

4.1. Problem (1.2)

As stated in the introduction, we are going to study perturbations of problem (1.1) which are more general than (1.3). They take the form

$$\begin{cases} \Delta v = a(x)v^p + h(x, v, \varepsilon) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We are imposing on the perturbation function h the following hypotheses, which will be termed throughout as hypotheses (H):

(a) $h : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, and $h(x, v, 0) = 0$ for $x \in \Omega$, $v \in \mathbb{R}^+$.
(b) $|h(x, v, \varepsilon) - h(x, v, \varepsilon')| \leqslant C|\varepsilon - \varepsilon'|d(x)^{-\gamma}v^p$, for $x \in \Omega$, $v \in \mathbb{R}^+$ and $\varepsilon, \varepsilon'$ small.
(c) The derivative of h with respect to v exists and $|h'(x, v, \varepsilon) - h'(x, v, \varepsilon')| \leqslant C|\varepsilon - \varepsilon'|d(x)^{-\gamma}(1 + v)^{p-1}$, for $x \in \Omega$, $v \in \mathbb{R}^+$ and $\varepsilon, \varepsilon'$ small.
(d) The second derivative of h with respect to v exists and $|h''(x, v, \varepsilon)| \leqslant C(\varepsilon)d(x)^{-\gamma}(1 + v)^{p-2}$ for $x \in \Omega$, $v \in \mathbb{R}^+$.

It is not hard to show that the perturbation $\lambda^{\frac{1}{p-1}} g\left(x, \lambda^{-\frac{1}{p-1}} v\right)$ in (1.3) verifies these hypotheses when we set $\varepsilon = \lambda^{-1}$. Thus Theorem 1 is a consequence of the following more general result.

Theorem 11. *Assume h verifies hypotheses* (H), *and $a(x)$ is a continuous function which satisfies* (A). *Then there exists $\varepsilon_0 > 0$ such that problem* (1.4) *has a unique positive solution v_ε for every ε such that $|\varepsilon| < \varepsilon_0$. Moreover,*

$$\lim_{\varepsilon \to 0} \sup_{\Omega} d(x)^\alpha |v_\varepsilon(x) - U(x)| = 0,$$

where $\alpha = (2 - \gamma)/(p - 1)$ and U is the unique solution to (1.4) *with $h \equiv 0$.*

Proof. For clarity we divide the proof in several steps.

Step 1: Setting $\alpha = (2 - \gamma)/(p - 1)$, every family of positive solutions $\{v_\varepsilon\}$ to (1.4) verifies $v_\varepsilon \to U$ in X_α, as $\varepsilon \to 0$.

Let v be a positive solution to (1.4). Then

$$\Delta v = a(x)v^p + h(x, v, \varepsilon) \geqslant (a(x) - C\varepsilon d(x)^{-\gamma})v^p \geqslant (1 - C_1C\varepsilon)a(x)v^p \tag{4.1}$$

in Ω, where C_1 is a positive constant. The method of sub and supersolutions and the uniqueness of U imply $v \leqslant (1 - C_1C\varepsilon)^{-\frac{1}{p-1}} U$. Thus $0 \leqslant v \leqslant Kd^{-\alpha}$ in Ω, for some $K > 0$ and by (4.1) $\Delta v \geqslant 0$ in Ω. Similarly we prove $\Delta v \leqslant (1 + C_2C\varepsilon)a(x)v^p$, from which it follows that $v \geqslant (1 + C_2C\varepsilon)^{-\frac{1}{p-1}} U$ and $\Delta v \leqslant Kd^{-\alpha-2}$ in Ω. So far we have proved $\sup_\Omega d(x)^\alpha |v(x)| < +\infty$ and $\sup_\Omega d(x)^{\alpha+2}|\Delta v(x)| < +\infty$. In order to conclude that $v \in X_\alpha$, we need the following important lemma:

Lemma 12. *Assume* $f \in L^\infty_{\mathrm{loc}}(\Omega)$ *satisfies* $\sup_\Omega d(x)^{\alpha+2}|f(x)| < +\infty$ *for some* $\alpha \in \mathbb{R}$. *Let* $u \in H^1_{\mathrm{loc}}(\Omega)$ *be a weak solution to* $\Delta u = f$, *with* $\sup_\Omega d(x)^\alpha |u(x)| < +\infty$. *Then there exists a constant* $C > 0$, *independent of* u *and* f *such that*

$$\sup_\Omega d(x)^{\alpha+1}|\partial_i u(x)| \leqslant C\left(\sup_\Omega d(x)^\alpha |u(x)| + \sup_\Omega d(x)^{\alpha+2}|f(x)|\right) \tag{4.2}$$

for every $1 \leqslant i \leqslant N$. *In particular,* $u \in X_\alpha$.

Proof. Fix $y \in \Omega$, and let $d = d(y)$. In the ball $B(y, d/3)$ we apply estimate (4.45) of Gilbarg and Trudinger [23] to obtain

$$d(y)|\partial_i u(y)| \leqslant C\left(\sup_{B(y,2d/3)} |u(x)| + d^2 \sup_{B(y,2d/3)} |f(x)|\right)$$

for a positive constant C independent of y, u and f. We now multiply this inequality by d^α, and take into account that $d(x) \geqslant d/3$ in $B(y, 2d/3)$. Hence

$$\begin{aligned} d(y)^{\alpha+1}|\partial_i u(y)| &\leqslant C\left(\sup_{B(y,2d/3)} d(x)^\alpha |u(x)| + \sup_{B(y,2d/3)} d(x)^{\alpha+2}|f(x)|\right) \\ &\leqslant C\left(\sup_\Omega d(x)^\alpha |u(x)| + \sup_\Omega d(x)^{\alpha+2}|f(x)|\right). \end{aligned}$$

Taking sup proves (4.2). Since $f \in L^\infty_{\mathrm{loc}}(\Omega)$, regularity theory shows (see the proof of Lemma 3) that $u \in C^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$. Thus our hypotheses on f and u together with (4.2) show that $u \in X_\alpha$. The proof is concluded. □

By Lemma 12 we deduce $v \in X_\alpha$. We now want to show that if $\{v_\varepsilon\}$ is a family of positive solutions to (1.4) for ε small, then $v_\varepsilon \to U$ in X_α. Note that since we already know

$$(1 + C_2 C\varepsilon)^{-\frac{1}{p-1}} U \leqslant v_\varepsilon \leqslant (1 - C_1 C\varepsilon)^{-\frac{1}{p-1}} U$$

it easily follows that $\sup_\Omega d(x)^\alpha |v_\varepsilon(x) - U(x)| \to 0$ as $\varepsilon \to 0$. Setting $w_\varepsilon = v_\varepsilon - U$, we obtain that $\Delta w_\varepsilon = f_\varepsilon$ in Ω, where $f_\varepsilon := a(x)(v_\varepsilon^p - U^p) + h(x, v_\varepsilon, \varepsilon)$.

We claim that $f_\varepsilon \to 0$ in Y_ε. Note that $d(x)^{\alpha+2}|h(x, v_\varepsilon, \varepsilon)| \leqslant C\varepsilon(\sup_\Omega d(x)^\alpha |v_\varepsilon (x)|)^p \to 0$ as $\varepsilon \to 0$. Also, thanks to Theorem 4(b), the Nemytskii operator of $z(x, v) = a(x)v^p$ is continuous from $Y_{\alpha-2}$ to Y_α, and hence $\sup_\Omega d(x)^{\alpha+2} a(x)|v_\varepsilon(x)^p - U(x)^p| \to 0$ as $\varepsilon \to 0$. This shows the claim. Applying Lemma 12 we obtain that $\|v_\varepsilon - U\|_X \to 0$ as $\varepsilon \to 0$, as was to be proved.

Step 2: The solution U to (1.4) with $\varepsilon = 0$ is nondegenerate.

Observe that by Corollary 5, hypotheses (A) on $a(x)$ and Theorem 8, the operator $T(u) = \Delta u - a(x)u^p$ is C^1 in a neighborhood of U. This allows us to consider the linearized equation:

$$\Delta\varphi = pa(x)U^{p-1}\varphi \quad \text{in } \Omega$$

for $\varphi \in X_\alpha$. We have to show that $\varphi \equiv 0$.

We claim that for every $t > 0$, $\varphi \leqslant tU$. Assume on the contrary that there exists $x_0 \in \Omega$ such that $\varphi(x_0) > tU(x_0)$, and define $\Omega_0 = \{\varphi > tU\} \cap B_r(x_0)$, where $r = d(x_0)/2$. Then

$$\Delta(\varphi - tU) > (p-1)ta(x)U^p \geqslant C\,tr^{-\alpha p-\gamma}$$

in Ω_0. The same argument as in the proof of Theorem 10 provides with $x_1 \in \partial\Omega_0$ such that $\varphi(x_1) > (1+C)tU(x_1)$ for some $C > 0$. Iterating the argument, we obtain a sequence of points x_n such that $\varphi(x_n) > (1+C)^n tU(x_n)$, which implies that φ/U is not bounded from above. But this is impossible, since $|\varphi| \leqslant Cd^{-\alpha}$ ($\varphi \in X_\alpha$) and $U \geqslant C'd^{-\alpha}$ in Ω (according to Theorem 8). Thus $\varphi \leqslant tU$, and since $t > 0$ is arbitrary, it follows that $\varphi \leqslant 0$. The same argument for $-\varphi$ implies $\varphi \equiv 0$.

Step 3: Construction of solutions.

We define the operator $\mathcal{F} : \mathbb{R} \times X_\alpha \to Y_\alpha$ by $\mathcal{F}(\varepsilon, v) := \Delta v - a(x)v^p - h(x, v, \varepsilon)$. By hypotheses (H) and Corollary 5 $\mathcal{F}$ is continuous and C^1 with respect to v in a neighborhood of $(0, U)$. Moreover, $\mathcal{F}(0, U) = 0$ and

$$\frac{\partial\mathcal{F}}{\partial v}(0, U) = \Delta - pa(x)U^{p-1}$$

is injective, as Step 2 shows. We need to show that it is an isomorphism. Thanks to the open mapping theorem, it will be enough to prove that it is surjective.

Take $m \in Y_\alpha$, and consider the problem

$$\Delta\varphi = pa(x)U^{p-1}\varphi + m(x) \quad \text{in } \Omega. \tag{4.3}$$

Note that $|m| \leqslant Ca(x)U^p$ for a certain positive constant C. Thus, it is not hard to show that $-CU$ and CU are ordered sub and supersolutions, respectively, for problem (4.3). Let us see that there exists at least a solution φ to (4.3) verifying $|\varphi| \leqslant Cd^{-\alpha}$. Indeed, define $\Omega_n = \{x \in \Omega : \ d(x) > 1/n\}$, and in Ω_n consider the problem

$$\begin{cases} \Delta\varphi = pa(x)U^{p-1}\varphi + m(x) & \text{in } \Omega_n, \\ \varphi = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{4.4}$$

It is well-known that problem (4.4) has a unique solution $\varphi_n \in H^1(\Omega_n) \cap H^2_{\mathrm{loc}}(\Omega_n)$, since $pa(x)U^{p-1} > 0$ (cf. [23]). Moreover, from the maximum principle we have

$-CU \leqslant \varphi_n \leqslant CU$ in Ω_n. Thus it is standard to conclude that for a subsequence, $\varphi_n \to \varphi$ in $C^1(\Omega)$, and φ is a weak solution to (4.3). It follows that $|\varphi| \leqslant Cd^{-\alpha}$ and $|\Delta\varphi| \leqslant Cd^{-\alpha-2}$, and Lemma 12 implies $\varphi \in X_\alpha$. Hence $\partial\mathcal{F}/\partial v(0,U)$ is surjective.

We are in a position now to apply the implicit function theorem, to obtain an $\varepsilon_0 > 0$ and a continuous function $u : (-\varepsilon_0, \varepsilon_0) \to X_\alpha$ such that $u(0) = U$, $\mathcal{F}(\varepsilon, u(\varepsilon)) = 0$ and the unique solution to $\mathcal{F}(\varepsilon, v) = 0$ in a neighborhood of $(0, U)$ is $u(\varepsilon)$. According to Step 1, $u_\varepsilon := u(\varepsilon)$ is the unique solution to (1.4) for ε small, provided that $u_\varepsilon = +\infty$ on $\partial\Omega$. Notice that by the continuity of $u(\varepsilon)$, for ε small we can assert that $\inf_\Omega d(x)^\alpha u_\varepsilon(x) > 0$, and hence $u_\varepsilon = +\infty$ on $\partial\Omega$. This proves the Theorem. $\square$

4.2. Problem (1.5)

We consider now problem (1.5). The proofs go essentially as before, with some minor technical changes. The scaling $v = u + \log\lambda$ reduces (1.5) into

$$\begin{cases} \Delta v = a(x)e^v + g(x, v - \log\lambda) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

As in Section 4.1, we can consider more general perturbations depending on a small parameter ε

$$\begin{cases} \Delta v = a(x)e^v + h(x, v, \varepsilon) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

The hypotheses for the perturbation term h in this case are similar to (H). However, the growth of h can be exponential now and so they take the form (hypotheses (H′))

(a) $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and $h(x, v, \varepsilon) = 0$ for $x \in \Omega$, $v \in \mathbb{R}$.
(b) $|h(x, v, \varepsilon) - h(x, v, \varepsilon')| \leqslant C|\varepsilon - \varepsilon'| d(x)^{-\gamma} e^v$, for $x \in \Omega$, $v \in \mathbb{R}$ and $\varepsilon, \varepsilon'$ small.
(c) The derivative of h with respect to v exists and $|h'(x, v, \varepsilon) - h'(x, v, \varepsilon')| \leqslant C|\varepsilon - \varepsilon'| d(x)^{-\gamma}(1 + e^v)$, for $x \in \Omega$, $v \in \mathbb{R}$ and $\varepsilon, \varepsilon'$ small.
(d) The second derivative of h with respect to v exists and $|h''(x, v, \varepsilon)| \leqslant C(\varepsilon) d(x)^{-\gamma}(1 + e^v)$ for $x \in \Omega$, $v \in \mathbb{R}$.

We leave to the reader to check that the perturbation in (4.5) satisfies these hypotheses, and so Theorem 2 is a consequence of

Theorem 13. *Assume h verifies hypotheses* (H′), *and $a(x)$ is a continuous function which satisfies* (A). *Then there exists $\varepsilon_0 > 0$ such that problem* (4.6) *has a unique positive solution v_ε for every ε such that $|\varepsilon| < \varepsilon_0$. Moreover*,

$$\lim_{\varepsilon \to 0} \sup_\Omega d(x)^{2-\gamma} |e^{v_\varepsilon(x)} - e^{V(x)}| = 0,$$

where V is the unique solution to (4.6) *with $h \equiv 0$.*

Proof. We first remark that, by changing $a(x)$ by $e^{-K}a(x)$ and V by $V+K$, we can always assume that $V \geqslant 1$.

As in Step 1 in the proof of Theorem 11, it can be shown that any solution v to (4.6) verifies

$$V - \log(1 + C_2 C\varepsilon) \leqslant v \leqslant V - \log(1 - C_1 C\varepsilon) \quad \text{in } \Omega. \tag{4.7}$$

In particular, we deduce that for every family of solutions $\{v_\varepsilon\}$, $\sup_\Omega |v_\varepsilon(x) - V(x)| \to 0$ as $\varepsilon \to 0$, and $\sup_\Omega d(x)^2 |\Delta v_\varepsilon(x) - \Delta V(x)| \to 0$ as $\varepsilon \to 0$. We can use Lemma 12 with $\alpha = 0$ to obtain that also $\sup_\Omega d(x)|\partial_i v_\varepsilon(x) - \partial_i V(x)| \to 0$ as $\varepsilon \to 0$, for every $1 \leqslant i \leqslant N$.

We now observe that problem (4.6) can be treated in the spaces X_α, Y_α for $\alpha = 2 - \gamma$ if we perform the change of variables $w = e^v$. With this change, the problem becomes

$$\begin{cases} \Delta w - \dfrac{|\nabla w|^2}{w} = a(x)w^2 + wh(x, \log w, \varepsilon) & \text{in } \Omega, \\ w = +\infty & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Note that (4.7) implies that $C_1 d(x)^{\gamma-2} \leqslant w(x) \leqslant C_2 d(x)^{\gamma-2}$ in Ω, for some positive constants C_1, C_2, and that $\lim_{\varepsilon\to 0} \sup_\Omega d(x)^{2-\gamma}|w_\varepsilon(x) - e^{V(x)}| = 0$ for every family of positive solutions $\{w_\varepsilon\}$ to (4.8). It is a consequence of the preceding discussion that also $\lim_{\varepsilon\to 0} \sup_\Omega d(x)^{\alpha+2}|\Delta w_\varepsilon(x) - \Delta e^{V(x)}| = 0$ and $\lim_{\varepsilon\to 0} \sup_\Omega d(x)^{\alpha+1}|\partial_i w_\varepsilon(x) - \partial_i e^{V(x)}| = 0$, for every $1 \leqslant i \leqslant N$. Thus we conclude that every family of positive solutions $\{w_\varepsilon\}$ to problem (4.8) verifies $w_\varepsilon \to W$ in X_α, where $W := e^V$.

As in Theorem 11, we introduce the operator $\mathcal{F} : \mathbb{R} \times D(Q) \to Y_\alpha$ defined by

$$\mathcal{F}(\varepsilon, w) := \Delta w - \frac{|\nabla w|^2}{w} - a(x)w^2 - wh(x, \log w, \varepsilon),$$

where $D(Q) = \{u \in X_\alpha : \ \inf_\Omega d(x)^\alpha |u(x)| > 0\}$. According to hypotheses (H′) and Corollary 7, $\mathcal{F}$ is C^1 in a neighborhood of $(0, W)$. We have to show that $\partial\mathcal{F}/\partial w(0, W)$ is an isomorphism, and the rest of the proof will go as in Theorem 11.

We first prove nondegeneracy of W. Let $\varphi \in X_\alpha$ solve the linearized equation

$$\Delta\varphi - \frac{2}{W}\nabla W \nabla\varphi + \frac{|\nabla W|^2}{W^2}\varphi = 2a(x)W\varphi \quad \text{in } \Omega.$$

We have to show that $\varphi \equiv 0$. Let $\varphi = W \log W \psi$. Observe that ψ is bounded in Ω and $\psi = 0$ on $\partial\Omega$. A tedious calculation leads to

$$V\Delta\psi + 2\nabla V \nabla\psi + a(x)e^V(1 - V)\psi = 0$$

in Ω, and since $V \geqslant 1$ the maximum principle implies $\psi \equiv 0$, thus $\varphi \equiv 0$.

To show that $\partial\mathcal{F}/\partial w(0, W)$ is surjective, we have to prove that for every $m \in Y_\alpha$ the equation

$$\Delta\varphi - \frac{2}{W}\nabla W \nabla\varphi + \frac{|\nabla W|^2}{W^2}\varphi = 2a(x)W\varphi + m(x) \quad \text{in } \Omega \tag{4.9}$$

has a solution $\varphi \in X_\alpha$. Setting $\psi = \varphi/W$, we now have to solve

$$\Delta\psi = a(x)W\psi + \frac{m(x)}{W(x)} \quad \text{in } \Omega \tag{4.10}$$

for a bounded function ψ. Notice that $|m(x)| \leqslant Ca(x)W(x)^2$ for a certain constant $C > 0$, and thus $\underline{u} = -C$ and $\overline{u} = C$ are sub and supersolutions to (4.10). As in Theorem 11, it follows that (4.10) has a bounded solution ψ. Also, $|\Delta\psi| \leqslant Cd^{-2}$, so Lemma 12 gives that $|\partial_i\psi| \leqslant Cd^{-1}$ for every $1 \leqslant i \leqslant N$. Recalling that $\varphi = \psi W$, it follows that $\varphi \in X_\alpha$, and is a solution to (4.9). This shows that $\partial\mathcal{F}/\partial w(0, W)$ is bijective, and by the open mapping theorem, it is an isomorphism. □

4.3. Some more general problems

We now consider perturbations of the problem (3.3) treated in Section 3. Since all the proofs will go as in §§4.1 and 4.2, we only show the important points.

More precisely, we are studying

$$\begin{cases} \Delta v = a(x)f(v) + h(x, v, \varepsilon) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where the function g verifies the hypotheses (H) in Section 4.1, and f satisfies

(a) $f : (0, +\infty) \to \mathbb{R}$ is C^1;
(b) $f(u)/u$ is increasing for $u > 0$;
(c) $\lim_{u\to+\infty} f'(u)/pu^{p-1} = 1$.

We note that these hypotheses imply the corresponding (a)–(c) in Section 3 (see also the remarks before Theorem 10). Thus, according to Theorem 10, problem (1.6) for $\varepsilon = 0$ has a unique positive solution Z. The analog of Theorems 11 and 13 is

Theorem 14. *Assume f verifies* (a)–(c), *h verifies hypotheses* (H), *and $a(x)$ is a continuous function which satisfies* (A). *Then there exists $\varepsilon_0 > 0$ such that problem* (1.6) *has a unique positive solution v_ε for every ε such that $|\varepsilon| < \varepsilon_0$. Moreover,*

$$\lim_{\varepsilon\to 0} \sup_{\Omega} d(x)^\alpha |v_\varepsilon(x) - Z(x)| = 0,$$

where $\alpha = (2-\gamma)/(p-1)$ and Z is the unique solution to (1.6) *with $h \equiv 0$.*

Sketch of the proof. We first prove that every family of positive solutions $\{v_\varepsilon\}$ verifies $v_\varepsilon \to Z$ as $\varepsilon \to 0$ in X_α.

Note that since $f(u)/u$ is increasing and $f(u) > 0$ in $[\min Z, +\infty)$, we have $f'(u)u/f(u) > 1$ in $[\min Z, +\infty)$. According to hypothesis (c) on f, there exists $\theta > 1$ such that $f'(u)u \geqslant \theta f(u)$ in $[\min Z, +\infty)$. This also shows that $f(u)/u^\theta$ is increasing in this range. Moreover, by hypotheses (H):

$$\Delta v_\varepsilon \leqslant a(x)f(v_\varepsilon) + C\varepsilon d(x)^{-\gamma} v_\varepsilon^p \leqslant (1 + C\varepsilon)a(x)f(v_\varepsilon)$$

in Ω. Since $f(u)/u^\theta$ is increasing, it follows that $(1 + C\varepsilon)f(v_\varepsilon) < f((1 + C\varepsilon)^{\frac{1}{\theta}} v_\varepsilon)$ and thus $(1 + C\varepsilon)^{\frac{1}{\theta}} v_\varepsilon$ is a supersolution to problem (1.6) with $h \equiv 0$. By the method of sub and supersolutions and the uniqueness given by Theorem 10 it follows that $(1+C\varepsilon)^{\frac{1}{\theta}} v_\varepsilon \geqslant Z$, and similarly $(1-C\varepsilon)^{\frac{1}{\theta}} v_\varepsilon \leqslant Z$. Hence $\sup_\Omega d(x)^\alpha |v_\varepsilon(x) - Z(x)| \to 0$ as $\varepsilon \to 0$, and consequently $v_\varepsilon \to Z$ in X_α (by simply reproducing the argument in Step 1 of Theorem 11).

We now prove that the solution Z is nondegenerate. Let $\varphi \in X_\alpha$ be a solution to the linearized equation

$$\Delta\varphi = a(x)f'(Z)\varphi \quad \text{in } \Omega$$

and fix $t > 0$. We claim that $\varphi \leqslant tZ$. Indeed, if we assume there exists $x_0 \in \Omega$ such that $\varphi(x_0) > tZ(x_0)$ and introduce the set $\Omega_0 = \{\varphi > tZ\} \cap B_r(x_0)$, where $r = d(x_0)/2$, we obtain

$$\Delta\varphi = a(x)f'(Z)\varphi > a(x)f'(Z)tZ \geqslant \theta t a(x)f(Z)$$

in Ω_0, thus $\Delta(\varphi - tZ) \geqslant (\theta - 1)ta(x)f(Z)$. Proceeding as in Theorem 10 (see also Step 2 in Theorem 11) we obtain that φ/Z is unbounded, which is impossible. Thus $\varphi \leqslant tZ$, and since $t > 0$ is arbitrary, we obtain $\varphi \leqslant 0$, and symmetrically $-\varphi \leqslant 0$, as we wanted to prove.

The rest of the proof goes exactly as in Theorem 11, and we are not giving the details. □

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